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On the Left Regular Representation of a Separable Locally Compact Group*

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For an arbitrary separable locally compact group G we exhibit a canonical Borel subset $\hat{G}_\Delta$ of the quasi-dual $\hat{G}$ of G (with the Mackey Borel structure), such that $\hat{G}_\Delta$ is a standard Borel space in the induced Borel structure, and such that the canonical measure for the left regular representation λ_G of G is concentrated on $\hat{G}_\Delta$. On the basis of this we discuss the (non-unimodular) "Plancherel theorem."

INTRODUCTION

Let G be a separable locally compact group and let $\chi\colon G \to \mathbb{R}_+^*$ be a continuous homomorphism. We call a (factor) representation π of G for χ-normal if π is associated with a χ-semicharacter on G (cf. [12]) and we set $\hat{G}_\chi$ to be the subspace of $\hat{G}$ (the quasi-dual of G) consisting of quasi-equivalence classes of χ-normal representations. One main purpose of this paper is to prove that $\hat{G}_\chi$ is a Borel subset of $\hat{G}$, when the latter is given the Mackey–Borel structure, and that $\hat{G}_\chi$ is a standard Borel space in the induced Borel structure (Theorem 3.2.2). For $\chi \equiv 1$ the space $\hat{G}_\chi$ is nothing but $\hat{G}_{\text{norm}}$, the space of normal representations of G, and in this special case the result has been obtained by Halpern [7].

The result just described (i.e., that $\hat{G}_\chi$ is a standard Borel subspace of $\hat{G}$) forms the basis for a brief discussion of a decomposition theory for χ-semitraces (Section 4). We show in particular that the canonical measure on $\hat{G}$ for the representation associated with a χ-semitrace is concentrated on $\hat{G}_\chi$. The most important application of this decomposition theory is certainly the

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instance where $\chi = \Delta$, the modular function of G, and where the semitrace in question is the Δ-semitrace associated with the left regular representation of G (whence the title of the paper). We have thus exhibited a *canonical* standard Borel subset $\hat{G}_\Delta$ of $\hat{G}$ on which the canonical measure for the left regular representation is concentrated. Incidentally, we believe that this is the first time a canonical standard Borel subset of the quasi-dual $\hat{G}$ of an arbitrary separable locally compact group G has been singled out in such a manner that this set may contain type III representations.

In the first part (Sections 1 and 2) of the paper we consider a C^*-dynamical system (A, G, α) and a continuous homomorphism $\chi: G \to \mathbb{R}_+^*$. Recall that a (G, χ)-trace f on A is a lower semicontinuous, semifinite trace, relatively invariant under the action of G with multiplier χ (cf. [13, §3]) and that f is a (G, χ)-character if the action of G in the von Neumann algebra associated with f is centrally ergodic. We call a representation π of A for (G, χ)-normal if π is associated with a (G, χ)-character. In Section 1 we equip the space $(A, G, \alpha)_\chi^\wedge$ of quasi-equivalence classes of (G, χ)-normal representations of A with a canonical standard Borel structure. The Borel space $(A, G, \alpha)_\chi^\wedge$ is then used as a base space for a decomposition of (G, χ)-traces into (G, χ)-characters (Section 2). In Sections 3 and 4 these results are then applied to the case where $A = C^*(G_0)$, $G_0 = \ker \chi$, and where α is the natural action of G in $C^*(G_0)$, and we obtain our results on χ-semitraces by "inducing" the results on (G, χ)-traces from the subgroup G_0 up to G. It is curious to observe that by using this technique we obtain a decomposition theory for χ-semitraces and in particular a "Plancherel formula" for non-unimodular groups exclusively with the use of decomposition theory for ordinary Hilbert algebras, that is, without using decomposition theory for left Hilbert algebras.

We end the paper with a discussion of certain questions related to the type of a semitrace (Section 5).

1. The (G, χ)-Normal Representations

1.1. Let (A, G, α) be a separable C^*-dynamical system, i.e., a triple of a separable C^*-algebra A, a separable locally compact group G and a homomorphism α from G into the group $\mathrm{Aut}(A)$ of $*$-isomorphisms of A, such that $s \to \alpha_s(a)$ is continuous for all $a \in A$. A^0 denotes the opposite C^*-algebra.

A representation π of A on the Hilbert space H_π is said to be quasi-invariant if the representation $\pi \circ \alpha_s$ is quasi-equivalent to π for all $s \in G$. If π is quasi-invariant, there exists an automorphism group $s \to \beta_s$ of $\mathfrak{A} = \pi(G)''$, such that $\beta_s \circ \pi = \pi \circ \alpha_s$. We say that π is G-factorial if β acts ergodically in the center $\mathfrak{Z}$ of $\mathfrak{A}$, that is, if $T \in \mathfrak{Z}$ and $\beta_s(T) = T$ for all $s \in G$ implies that T is a scalar.

Let $\chi: G \to \mathbb{R}_+^*$ be a continuous homomorphism. A lower semicontinuous, semifinite trace f is called a (G, χ)-trace if it is relatively invariant under α with multiplier χ, i.e., if $f(\alpha_s(a)) = \chi(s) f(a)$ for all $a \in A^+$, $s \in G$ (cf. [13, § 3]).

If f is a (G, χ)-trace, we associate as usual the objects H_f, λ_f, ρ_f, U_f, V_f, J_f, ω_f with f (cf. [1, § 6]; the notation loc. cit. for ω_f is t_f). We also associate a representation v_f of G on H_f by

$$v_f(s)\dot{a} = \chi(s)^{-1/2} (\alpha_s(a))^{\cdot},$$

$s \in G$, $a \in \mathfrak{n}_f$ (cf. [13, § 3]). We have $v_f(s) \lambda_f(a) v_f(s^{-1}) = \lambda_f(\alpha_s(a))$ and $v_f(s) \rho_f(a) v_f(s^{-1}) = \rho_f(\alpha_s(s))$. In particular λ_f and ρ_f are quasi-invariant. Let $s \to \beta_f(s)$ be the automorphism group of U_f implemented by v_f (i.e., $\beta_f(s)(\lambda_f(a)) = \lambda_f(\alpha_s(a))$). Also set $\mathfrak{Z}_f = U_f \cap V_f$ and let $\mathfrak{Z}_f^{(\beta_f)}$ be the fixed point algebra of β_f in $\mathfrak{Z}_f$. We say that f is a (G, χ)-character if λ_f is G-factorial (i.e., if $\mathfrak{Z}_f^{(\beta_f)} = \mathbb{C}I$).

1.2. Let π be a quasi-invariant representation of A and let $s \to \beta_s$ be the automorphism group of $\mathfrak{A} = \pi(A)''$, such that $\beta_s \circ \pi = \pi \circ \alpha_s$.

LEMMA 1.2.1. *Let ω_1 and ω_2 be faithful, normal semifinite traces on $\mathfrak{A}$, relatively invariant under β with multiplier χ (i.e., $\omega_j \circ \beta_s = \chi(s)\, \omega_j$, $j = 1, 2$). If π is G-factorial, then ω_1 and ω_2 are proportional.*

Proof. This lemma is the analog of [2, Corollaire, p. 92] and is proved along the same lines of reasoning.

DEFINITION 1.2.2. The quasi-invariant representation π is called (G, χ)-normal if

(i) π is G-factorial,

(ii) $\mathfrak{A} = \pi(A)''$ admits a faithful, normal, semifinite trace ω, relatively invariant under β with multiplier χ,

(iii) there exists $a \in A^+$, such that $0 < \omega(\pi(a)) < +\infty$.

(Lemma 1.2.1 shows that there is essentially one such choice of ω.)

LEMMA 1.2.3. *Assume that π is (G, χ)-normal and let ω be a trace as described in Definition* 1.2.2(ii). *Then (π, ω) is a trace class representation* [1, *Définition* 6.6.1] *and $\omega \circ \pi \mid A^+ = f$ is a (G, χ)-character. Moreover, (π, ω) and (λ_f, ω_f) are quasi-equivalent.*

Proof. It is easily seen that (π, ω) is a trace class representation (cf.

proof of [1, Proposition, p. 127]). The rest of the lemma is contained in [1, § 6].

LEMMA 1.2.4. *Let f and f_1 be two (G, χ)-characters. If λ_f and λ_{f_1} are not quasi-equivalent, then they are disjoint.*

Proof. Define the representation $\bar{\lambda}_f$ on $\bar{H}_f = L^2(G, H_f)$ by $\bar{\lambda}_f(a)\, g(t) = \lambda_f \circ \alpha_t^{-1}(a)\, g(t)$, and define $V: \bar{H}_f \to \bar{H}_f$ by $Vg(t) = v_f(t)\, g(t)$. Then V is a unitary and it carries the representation $\bar{\lambda}_f$ to $a \to \lambda_f(a) \otimes I$ if we identify $L^2(G, H_f)$ with $H_f \otimes L^2(G)$. This shows that $\bar{\lambda}_f$ is quasi-equivalent to λ_f. Let then π be a non-zero subrepresentation of λ_f corresponding to the projection E in $U_f' = V_f$. Define the representation $\bar{\pi}$ on $L^2(G, E)$ (we identify E and the cosed subspace it defines in H_f) given by $\bar{\pi}(a)\, g(t) = \pi \circ \alpha_t^{-1}(a)\, g(t)$. $\bar{\pi}$ can be considered as a subrepresentation of $\bar{\lambda}_f$ corresponding to the subspace $\bar{E} = L^2(G, E) \subset L^2(G, H_f)$. Let $s \to u(s)$ be the representation $u(s)\, g(t) = g(s^{-1}t)$ on $\bar{H}_f$. We have $\bar{\lambda}_f(\alpha_s(a)) = u(s)\, \bar{\lambda}_f(a)\, u(s^{-1})$. Now since $\bar{E}$ clearly is $u(s)$-invariant and since also $u(s)\, \mathfrak{A}\, u(s^{-1}) = \mathfrak{A}$ and therefore $u(s)\, \mathfrak{A}' u(s^{-1}) = \mathfrak{A}'$, where $\mathfrak{A} = \bar{\lambda}_f(A)''$, we find that the central support of $\bar{E}$ in $\mathfrak{A}$ is $u(s)$-invariant. But then this central support is $\bar{H}_f$ since $\bar{\lambda}_f \approx \lambda_f$ is G-factorial. We have thus shown that $\bar{\pi} \approx \bar{\lambda}_f \approx \lambda_f$.

Suppose then that λ_f and λ_{f_1} are not disjoint. Then there exist subrepresentations π and π_1 of λ_f and λ_{f_1}, respectively, such that π and π_1 are equivalent and therefore $\lambda_f \approx \lambda_{f_1}$. This ends the proof of the lemma.

1.3. If π is a (G, χ)-normal representation of A and if π' is quasi-equivalent to π, then π' is (G, χ)-normal.

We define $(A, G, \alpha)_\chi^{\check{}}$ to be the set of equivalence classes (under "quasi-equivalence") of (G, χ)-normal representations.

A (G, χ)-character f gives rise to an element in $(A, G, \alpha)_\chi^{\check{}}$, namely, to the class of λ_f. Conversely, if π is a (G, χ)-normal representation, there exists a (G, χ)-character f such that π is quasi-equivalent to λ_f (Lemma 1.2.3) and if f_1 is another (G, χ)-character, such that λ_f and λ_{f_1} are quasi-equivalent, then f and f_1 are proportional (Lemma 1.2.1). In other words, there is a bijection between the space $(A, G, \alpha)_\chi^{\check{}}$ and the set of (G, χ)-characters defined up to a scalar.

1.4. If f is a (G, χ)-character, we denote by $[f]$ the element in $(A, G, \alpha)_\chi^{\check{}}$ determined by f (that is, $[f]$ is the class of λ_f).

If X is a Borel space and $\xi \to f_\xi$, $\xi \in X$, is a family of (G, χ)-traces, we say that $\xi \to f_\xi$ is a Borel field of (G, χ)-traces if $\xi \to f_\xi(a)$ is a Borel function for each $a \in A^+$.

THEOREM 1.4.1. *On $(A, G, \alpha)_\chi^{\check{}}$ there exists a standard Borel structure,* *uniquely determined by the following property: If X is a standard Borel space and $\xi \to f_\xi$ is a Borel field of (G, χ)-characters, then $\xi \to [f_\xi]: X \to (A, G, \alpha)\hat{}_\chi$ is a Borel function.*

Proof. We prove this in several steps (Sections 1.5–1.8):

1.5.

LEMMA 1.5.1. *There exists a countable subset S of A^+ with the following property: For each non-zero lower semicontinuous, semifinite trace f on A there exists $a \in S$, such that* $0 < f(a) < +\infty$.

Proof. Let $\mathbb{J}$ be a countable family of closed ideals in A, such that $\{\hat{J} \mid J \in \mathbb{J}\}$ is a basis for the topology on $\mathrm{Prim}(A)$ [1, 3.3.4]. Let S_J be a countable dense subset in $K(J)^+$, the positive part of the Pedersen ideal of J [11, 5.6, p. 75] and set $S = \bigcup_{J \in \mathbb{J}} S_J$. We claim that S has the required properties. Let f be a trace as described and let $J = \bar{\mathfrak{m}}_f$. Let J_n be a sequence in $\mathbb{J}$ with $\bigcup_n \hat{J}_n = \hat{J}$, i.e., with J equal to the closure of all finite sums $\sum a_n$, $a_n \in J_n$. We have $J_n \subset J$ and therefore $K(J_n) \subset K(J) \subset \mathfrak{m}_f$ by the minimally of $K(J)$. Clearly f cannot vanish on all of $K(J_n)^+$ unless it is 0, and f cannot vanish on S_{J_n} unless it vanishes $K(J_n)^+$. This means that there exists $n \in \mathbb{N}$, such that f does not vanish on S_{J_n}. This proves the lemma since f is clearly finite on S_{J_n}.

Remark 1.5.2.. The lemma just proved is a sharpened version of Lemma 1 in [7]. Here we take a more convenient route and exploit the nice properties of the Pedersen ideal, cf. [11, 5.6, p. 175].

1.6. Let $\mathfrak{a}$ be a selfadjoint two-sided α-invariant ideal in A. By $B(\mathfrak{a})$ we denote the set of all bitraces on $\mathfrak{a}$ [1, 6.2.1. Définition, p. 117] and by $B_\chi(\mathfrak{a})$ the set of all relatively invariant bitraces on A with multiplier χ. This means that $b \in B(\mathfrak{a})$ is in $B_\chi(\mathfrak{a})$ if $b(\alpha_s(x), \alpha_s(y)) = \chi(s)\, b(x, y)$, $s \in G$, $x, y \in \mathfrak{a}$. For an element $b \in B_\chi(\mathfrak{a})$ we define the objects H_b, λ_b, ρ_b, U_b, V_b, J_b, ω_b (cf. [1, 6.2]; ω_b is called t_b loc. cit.). We also define the representation v_b of G on H_b by $v_b(s)\dot{a} = \chi(s)^{-1/2} (\alpha_s(a))^{\cdot}$. In this fashion $v_b(s)\, \lambda_b(x)\, v_b(s^{-1}) = \lambda_b(\alpha_s(x))$.

$B(\mathfrak{a})$ is equipped with the topology of pointwise convergence on $\mathfrak{a} \times \mathfrak{a}$ and with the underlying topological Borel structure. Clearly $B_\chi(\mathfrak{a})$ is closed in $B(\mathfrak{a})$.

LEMMA 1.6.1. *There exists Borel maps $b \to L_b$, $b \to R_b$, $b \to V_b$ from $B_\chi(\mathfrak{a})$ into* $\mathrm{Rep}(A)$, $\mathrm{Rep}(A^0)$, $\mathrm{Rep}(G)$, *respectively, with the following properties:* (i) *for each $b \in B_\chi(\mathfrak{a})$*, $\dim L_b = \dim R_b = \dim V_b = \dim H_b = n_b$

and (ii) *for each* $b \in B_\chi(\mathfrak{a})$ *there exists a unitary* $U: H_b \to H_{n_b}$ *carrying* λ_b *into* L_b, ρ_b *into* R_b *and* v_b *into* v_b (*for* Rep(A), H_{n_b} *etc.*, cf. [1, 3.5]).

Proof. This is an extension of [6, Lemma 2, p. 18]. The proof of the latter result can easily be adapted to give a proof of this lemma.

We denote by $B'_\chi(\mathfrak{a})$ the set of elements $b \in B_\chi(\mathfrak{a})$, such that the quasi-invariant representation λ_b of A is G-factorial.

LEMMA 1.6.2. *$B'_\chi(\mathfrak{a})$ is a Borel subset of $B_\chi(\mathfrak{a})$.*

Proof. (a) First let us observe that λ_b is G-factorial if and only if the von Neumann algebra generated by $\lambda_b(A)$, $\rho_b(A)$, $v_b(G)$ is $B(H_b)$. In fact, the commutant of this algebra is the intersection of $\lambda_b(A)''$, $\rho_b(A)''$ and $v_b(G)'$ and this is precisely the set of elements in the center $\mathfrak{Z}$ of $\lambda_b(A)''$ commuting with $v_b(s)$ for all $s \in G$.

(b) The subset Z_n of $B_\chi(\mathfrak{a})$ consisting of elements with $\dim H_b = n$, $n = 1, 2, \ldots, +\infty$, is a Borel subset (Lemma 1.6.1). Set $Z'_n = Z_n \cap B'_\chi(\mathfrak{a})$. It clearly suffices to show that Z'_n is Borel in Z_n.

(c) Let $b \to L_b$, $b \to R_b$, $b \to V_b$ be Borel maps on $B_\chi(\mathfrak{a})$ as described in Lemma 1.6.1. From the remarks in (a) it follows that an element b in Z_n is in Z'_n if and only if the von Neumann algebra generated by $L_b(A)$, $R_b(A)$, $V_b(G)$ is $B(H_n)$.

(d) For each $m \in \mathbb{N}$ let $\mathscr{P}_m$ be the set of polynomials in $3m$ non-commuting variables $X_1, \ldots, X_m$, $Y_1, \ldots, Y_m$, $Z_1, \ldots, Z_m$ and with integer coefficients. Set $\mathscr{P} = \bigcup_m \mathscr{P}_m$. Clearly, $\mathscr{P}$ is a countable set. For each $b \in B_\chi(\mathfrak{a})$ define $M(b)$ to be the set of elements in $B(H_{n_b})$ ($n_b = \dim H_b$) of the form $P(L_b(x_1), \ldots, L_b(x_m),\ R_b(y_1), \ldots, R_b(y_m),\ V_b(z_1), \ldots, V_b(z_m))$ for some $m \in \mathbb{N}$, $P \in \mathscr{P}_m$ and some elements $x_1, \ldots, x_m$, $y_1, \ldots, y_m \in A$, $z_1, \ldots, z_m \in C^*(G)$. It is easily seen that $M(b)$ is in fact a subalgebra of $B(H_{n_b})$. Clearly $M(b)$ is weakly dense in the von Neumann algebra generated by $L_b(A)$, $R_b(A)$, $V_b(G)$ and therefore we find, using Kaplansky's density theorem, that an element $b \in B_\chi(\mathfrak{a})$ is in $B'_\chi(\mathfrak{a})$ if and only if $M(b)_1 = M(b) \cap B(H_{n_b})_1$ is strongly dense in $B(H_{n_b})_1$.

(e) Let $a_1, a_2, \ldots$ be a dense sequence in A and let $c_1, c_2, \ldots$ be a dense sequence in $C^*(G)$. For $P \in \mathscr{P}_m$ we define the function $P_n: Z_n \to B(H_n)$ by

$$P_n(b) = P(L_b(a_1), \ldots, L_b(a_m),\ R_b(a_1), \ldots, R_b(a_m),\ V_b(c_1), \ldots, V_b(c_m)).$$

Clearly P_n is a Borel function. Let n be fixed and let $T_1, T_2, \ldots$ be a strongly dense sequence in $B(H_n)_1$ and let d be a distance function in $B(H_n)_1$ defining the strong topology. We set

$$Q_{P,p,q} = \{b \in Z_n \mid \|P_n(b)\| < 1 \text{ and } d(P_n(b), T_p) < 1/q\}.$$

$Q_{P,p,q}$ is a Borel subset of Z_n and we claim that $B'_\chi(\mathfrak{a}) = \bigcap_{p,q} \bigcup_{P\in\mathscr{P}} Q_{P,p,q}$. We have to prove that if $b \in Z_n$, then $M(b)_1$ is strongly dense in $B(H_n)_1$ if and only if there for each $p, q \in \mathbb{N}$ exists $P \in \mathscr{P}$ such that $b \in Q_{P,p,q}$. In fact, assume $M(b)_1$ is strongly dense in $B(H_n)_1$ and let $p, q \in \mathbb{N}$ be given. There exists $P \in \mathscr{P}_m$ and $x_1,..., x_m, y_1,..., y_m \in A$, $z_1,..., z_m \in C^*(G)$, such that

$$\|P(L_b(x_1),..., L_b(x_m), R_b(y_1),..., R_b(y_m), V_b(z_1),..., V_b(z_m))\| < 1$$

and

$$d(P(L_b(x_1),..., L_b(x_m), R_b(y_1),..., R_b(y_m), V_b(z_1),..., V_b(z_m)), T_p) < 1/q.$$

But then we can find $a_{i_1},..., a_{i_m}, a_{i_{m+1}},..., a_{i_{2m}}$ with $i_1 < \cdots < i_m < i_{m+1} < \cdots < i_{2m}$ and $c_{j_1},..., c_{j_m}$ with $j_1 < \cdots < j_m$, such that

$$\|P(L_b(a_{i_1}),..., L_b(a_{i_m}), R_b(a_{i_{m+1}}),..., R_b(a_{i_{2m}}), V_b(c_{j_1}),..., V_b(c_{j_m}))\| < 1$$

and

$$d(P(L_b(a_{i_1}),..., L_b(a_{i_m}), R_b(a_{i_{m+1}}),..., R_b(a_{i_{2m}}), V_b(c_{j_1}),..., V_b(c_{j_m}), T_p) < 1/q$$

Then setting $m' = \max\{i_{2m}, j_m\}$ and defining $P' \in \mathscr{P}_{m'}$ by

$$\begin{aligned} &P'(X_1,..., X_{m'}, Y_1,..., Y_{m'}, Z_1,..., Z_{m'}) \\ &\quad = P(X_{i_1},..., X_{i_m}, Y_{i_{m+1}},..., Y_{i_{2m}}, Z_{j_1},..., Z_{j_m}) \end{aligned}$$

we find that

$$P'_n(b) = P(L_b(a_{i_1}),..., L_b(a_{i_m}), R_b(a_{i_{m+1}}),..., R_b(a_{i_{2m}}),\ V_b(c_{j_1}),..., V_b(c_{j_m})).$$

But then $b \in Q_{P',p,q}$. Conversely, if $b \in \bigcap_{p,q} \bigcup_{P\in\mathscr{P}} Q_{P,p,q}$, then clearly $M(b)_1$ is strongly dense in $B(H_n)_1$. This proves the lemma.

LEMMA 1.6.3. *Let $a \in A^+$ and let $\mathfrak{a} = \mathfrak{a}(a)$ be the smallest α-invariant ideal in A containing a and let $B_\chi(\mathfrak{a})_1$ be the set of elements b in $B_\chi(\mathfrak{a})$, such that $b(a, a) = 1$. $B_\chi(\mathfrak{a})_1$ is a polish space (in the induced topology).*

Proof. If G is trivial, this is Proposition 2 in [7]. We can obtain a proof of our lemma by modifying the proof loc. cit. We shall indicate the main points where a change is needed. Let B be the set of functions $b: \mathfrak{a} \times \mathfrak{a} \to \mathbb{C}$ satisfying properties (i)–(iv) in the definition of bitraces [1, 6.2.1. Définition, p. 117] and furthermore (vi) $b(a, a) = 1$ and (vii)

$$b(\alpha_s(x), \alpha_s(y)) = \chi(s)\, b(x, y).$$

Let A_e, κ be as loc. cit. The ideal $\mathfrak{a} = \mathfrak{a}(a)$ is given by

$$\mathfrak{a} = \left\{ \sum_{i=1}^{m} x_i \alpha_{s_i}(a)\, y_i \mid x_i, y_i \in A_e, s_i \in G \right\}.$$

As loc. cit. one shows that

$$b\left(\sum_{i=1}^{m} x_i \alpha_{s_i}(a)\, y_i, \sum_{i=1}^{m} x'_i \alpha_{s'_i}(a)\, y'_i\right)$$
$$\leqslant \kappa^4 \sum_{i,j=1}^{m} \chi(s_i)\, \chi(s'_j)\, \|x_i\|^2\, \|x'_j\|^2\, \|y_i\|^2\, \|y'_j\|^2$$

for $b \in B$ and therefore $\{|b(x, y)| \mid b \in B\}$ is bounded for $x, y \in \mathfrak{a}$ fixed. Let $\mathscr{A}$ be a countable dense subset of A_e, let D be a countable, dense subset of G and let $(u_i)_{i \in \mathbb{N}}$ be an enumeration of the countable, dense subset C of $\mathfrak{a}$ given by

$$C = \left\{ \sum_{i=1}^{m} x_i \alpha_{s_i}(a)\, y_i \mid x_i, y_i \in \mathscr{A}, s_i \in D \right\}.$$

With these ingredients the proof of Lemma 1.6.3 can be finished following the approach of [7, pp. 134–135].

COROLLARY 1.6.4. *For $a \in A^+$ and $\mathfrak{a} = \mathfrak{a}(a)$ the set $B'_\chi(\mathfrak{a})_1$ of elements $b \in B_\chi(\mathfrak{a})$, such that $b(a, a) = 1$ and such that λ_b is G-factorial, is a Borel subset of $B_\chi(\mathfrak{a})$ and it is standard in the induced Borel structure.*

Proof. This follows from Lemmas 1.6.2 and 1.6.3.

For $b \in B'_\chi(\mathfrak{a})$, let $[\lambda_b]$ denote the canonical image of λ_b in $(A, G, \alpha)^\frown_\chi$.

LEMMA 1.6.5. *With the notation from Corollary 1.6.4 we have: The map $B'_\chi(\mathfrak{a})_1 \to (A, G, \alpha)^\frown_\chi \colon b \to [\lambda_b]$ is injective.*

Proof. This is straightforward using [1, § 6].

1.7. Let then $(a_n)_{n \in \mathbb{N}}$ be a sequence in A^+, such that the set $S = \{a_n \mid n \in \mathbb{N}\}$ has the property described in Lemma 1.5.1. We set $B_n = B'_\chi(\mathfrak{a}(a_n))_1$ with its topological Borel structure, which is standard (Corollary 1.6.4). Let $\mathscr{B}_n$ be the image of B_n in $(A, G, \alpha)^\frown_\chi$ by the map $b \to [\lambda_b]$.

LEMMA 1.7.1. *We have $\bigcup_{n \in \mathbb{N}} \mathscr{B}_n = (A, G, \alpha)^\frown_\chi$.*

Proof. Let f be a (G, χ)-character on A and pick $n \in \mathbb{N}$, such that $0 < f(a_n) < +\infty$. Defining $b(x, y) = \lambda^{-1} f(y^* x)$ for $x, y \in \mathfrak{a} = \mathfrak{a}(a_n)$, where

$\lambda = f(a_n^* a_n)$ we have that $b \in B'_\chi(\mathfrak{a})_1 = B_n$ and $[\lambda_b] = [f]$, cf. [7, pp. 136–137].

LEMMA 1.7.2, *Let $j, k \in \mathbb{N}$. The set $\{b \in B_j \mid [\lambda_b] \notin \mathscr{B}_k\}$ is a Borel subset in B_j.*

Proof. Let $b \to L_b: B_\chi(\mathfrak{a}(a_l)) \to \operatorname{Rep}(A)$, $l = j, k$, be Borel functions as described in Lemma 1.6.1. The complement of $\{b \in B_j \mid [\lambda_b] \notin \mathscr{B}_k\}$ is $\{b \in B_j \mid \exists b' \in B_k: L_b^j \simeq L_{b'}^k\}$. For $n = 1, 2, \ldots, +\infty$, let B_l^n, $l = j, k$, be the set of elements b in B_l with $\dim \lambda_b = n$. B_l^n is a Borel subset of B_l (Lemma 1.6.1). Therefore, to prove the lemma it suffices to show that $\{b \in B_j^n \mid \exists b' \in B_k^n: L_b^j \simeq L_{b'}^k\}$ is a Borel subset of B_j^n for each n. Let $\mathscr{I}: \operatorname{Rep}_n(A) \times \operatorname{Rep}_n(A) \to \mathbb{Z}$ be the map which to a pair (π_1, π_2) associates its intertwining number. From [1, 3.7.2 Lemma, p. 76] it follows that $\mathscr{I}$ is a Borel function. It follows from [1, 5.2.1 Proposition] and Lemma 1.2.4 that $L_b^j \simeq L_b^k$ if and only if $\mathscr{I}(L_b^j, L_b^k) > 0$. Set $T = \{(b, b') \in B_j^n \times B_k^n \mid \mathscr{I}(L_b^j, L_b^k) > 0\}$. T is a Borel subset of $B_j^n \times B_k^n$. Let $c: B_j^n \times B_k^n \to B_j^n$ be the projection onto the first coordinate. When restricted to T the map c is injective (Lemma 1.6.5) and its image is $\{b \in B_j^n \mid \exists b' \in B_k^n: \mathscr{I}(L_b^j, L_b^k) > 0\}$. This set is a Borel set since T and B_j^n are standard Borel spaces.

For $b \in B_j$, let f_b be the (G, χ)-character $\omega_b \circ \lambda_b \mid A^+$.

LEMMA 1.7.3. *The function $b \to f_b$ on B_j is a Borel field of (G, χ)-characters (cf. 1.4).*

Proof. Set $\mathfrak{a} = \mathfrak{a}(a_j)$ and let $(u_n)_{n \in \mathbb{N}}$ be an approximating unit in $\mathfrak{a}$ contained in $\bar{\mathfrak{a}}$. Set $g(b) = f_b(a^* a)$, $g_n(b) = f_b((u_n a)^* u_n a)$. Since $\lambda_b(u_n) \to I$ strongly, we find $g(b) = f_b(a^* a) = \omega_b(\lambda_b(a)^* \lambda_b(a)) \leqslant \liminf \omega_b(\lambda_b(a)^* \lambda_b(u_n)^* \lambda_b(u_n) \lambda_b(a)) \leqslant \omega_b(\lambda_b(a)^* \lambda_b(a))$ from which $g_n(b) \to g(b)$. Now $g_n(b) = b(u_n a, u_n a)$, and therefore $b \to g_n(b)$ is a Borel function on B_j; hence g is a Borel function on B_j. This proves the lemma.

1.8. We shall now end the proof of Theorem 1.4.1. We use the notation from Section 1.7.

We set $\mathscr{C}_j = \bigcap_{k=1}^{j-1} (\mathscr{B}_j \backslash \mathscr{B}_k)$, $j = 1, 2, \ldots$. Clearly $(A, G, \alpha)_\chi^\wedge = \bigcup_{j=1}^\infty \mathscr{C}_j$ (Lemma 1.7.1) and the union is disjoint. Let C_j be the subset in B_j which is the inverse image of $\mathscr{C}_j \subset \mathscr{B}_j$ by the map $b \to [\lambda_b]$. We have $C_j = \bigcap_{k=1}^{j-1} \{b \in B_j \mid [\lambda_b] \notin \mathscr{B}_k\}$ and thus C_j is a Borel subset of B_j (Lemma 1.7.2). In particular C_j is a standard Borel space in the induced Borel structure.

Let us then show the uniqueness of the Borel structure described in Theorem 1.4.1: It follows from Lemma 1.7.3 and the above remarks that the function $b \to f_b$ on C_j is a Borel field of (G, χ)-characters on the standard Borel space C_j. Therefore the map $b \to f_b$ is a Borel map from C_j to

$(A, G, \alpha)_\chi^{\hat{}}$; it is injective and its image is $\mathscr{C}_j$. But then $\mathscr{C}_j$ is a Borel subset of $(A, G, \alpha)_\chi^{\hat{}}$ and the map $b \to [f_b] = [\lambda_b]$ establishes a Borel isomorphism between C_j and $\mathscr{C}_j$. This shows the uniqueness.

We then show the existence of a Borel structure as described in Theorem 1.4.1. We equip $\mathscr{C}_j$ with the standard Borel structure it gets by transporting the Borel structure on C_j via the bijection $b \to [\lambda_b]: C_j \to \mathscr{C}_j$ and give $(A, G, \alpha)_\chi^{\hat{}}$ the Borel structure which is the sum of the Borel structures on the $\mathscr{C}_j$'s. This makes sense since $(A, G, \alpha)_\chi^{\hat{}}$ is the disjoint union of the sets $\mathscr{C}_j$, $j = 1, 2, \ldots$. In this fashion $(A, G, \alpha)_\chi^{\hat{}}$ is a standard Borel space. We have left to show that this Borel structure has the property described in Theorem 1.4.1: Let X be a standard Borel space and let $\xi \to f_\xi$ be a Borel field of (G, χ)-characters. We set $X'_j = \{\xi \in X \mid 0 < f_\xi(a_j) < +\infty\}$. We have $X = \bigcup_{j \in \mathbb{N}} X'_j$ and X'_j is a Borel set. Set $X_j = \bigcap_{k=1}^{j-1} (X'_j \backslash X'_k)$. Then X_j is a Borel set and $X = \bigcup_{j \in \mathbb{N}} X_j$ as a disjoint union. It follows that it suffices to show that $\xi \to [f_\xi]: X_j \to (A, G, \alpha)_\chi^{\hat{}}$ is a Borel function. We define the Borel function $k_j: X_j \to]0, +\infty[$ by $k_j(\xi) = f_\xi(a_j^* a_j)$, and define the bitrace b_ξ on $\mathfrak{a} = \mathfrak{a}(a_j)$ by $b_\xi(x, y) = k_j(\xi)^{-1} f_\xi(y^* x)$. The map $\xi \to b_\xi(x, x)$ is a Borel function for all $x \in \mathfrak{a}$ and therefore $\xi \to b_\xi(x, y)$ is a Borel function for all $x, y \in \mathfrak{a}$ by polarisation. But then the map $X_j \to C_j: \xi \to b_\xi$ is a Borel function; hence $\xi \to [b_\xi] = [f_\xi]: X_j \to (A, G, \alpha)_\chi^{\hat{}}$ is a Borel function. This ends the proof of Theorem 1.4.1.

1.9. In the course of proof of Theorem 1.4.1 we have obtained the following result (cf. Lemma 1.7.3).

PROPOSITION 1.9.1. *There exists a Borel field $\xi \to f_\xi$ of (G, χ)-characters on $(A, G, \alpha)_\chi^{\hat{}}$ such that $[f_\xi] = \xi$ for all $\xi \in (A, G, \alpha)_\chi^{\hat{}}$. All other such fields $\xi \to f'_\xi$ have the form $f'_\xi = k(\xi) f_\xi$ for a unique Borel function $k: (A, G, \alpha)_\chi^{\hat{}} \to]0, +\infty[$.*

We have also proved (cf. Lemma 1.6.1).

PROPOSITION 1.9.2. *There exists a Borel map $\xi \to L_\xi: (A, G, \alpha)_\chi^{\hat{}} \to \mathrm{Rep}(A)$ such that L_ξ is equivalent to λ_f, where f is a (G, χ)-character with $\lambda_f \in \xi$.*

Remark 1.9.3. It follows from Proposition 1.9.2 that if G (and therefore χ) is trivial, then the Borel structure defined here on $\hat{A}_{\mathrm{norm}} = (A, G, \alpha)_\chi^{\hat{}}$ is identical to the Mackey–Borel structure. In fact, the map $\xi \to L_\xi$ gives rise to a bijective Borel map between two standard Borel spaces (cf. [7]) and therefore to a Borel isomorphism.

2. Decomposition of (G,χ)-Traces

Let (A, G, α) be a separable C^*-dynamical system and let $\chi\colon G \to \mathbb{R}_+^*$ be a continuous homomorphism. Set $X = (A, G, \alpha)_\chi^\wedge$ (equipped with the standard Borel structure described in Theorem 1.4.1) and let $\xi \to f_\xi$ be a Borel field of (G,χ)-characters on X such that $[\lambda_{f_\xi}] = \xi$ for all $\xi \in X$ (Proposition 1.9.1).

THEOREM 2.1. *For each (G,χ)-trace f on A there exists a measure μ on X uniquely determined by the following property: The field $\xi \to H_{f_\xi}$ can be given a μ-measurable structure such that there exists an isomorphism from H_f onto $\int_X^\oplus H_{f_\xi}\, d\mu(\xi)$ setting up the following correspondence:*

$$\lambda_f \to \int_X^\oplus \lambda_{f_\xi}\, d\mu(\xi), \qquad \rho_f \to \int_X^\oplus \rho_{f_\xi}\, d\mu(\xi),$$

$$U_f \to \int_X^\oplus U_{f_\xi}\, d\mu(\xi), \qquad V_f \to \int_X^\oplus V_{f_\xi}\, d\mu(\xi),$$

$$J_f \to \int_X^\oplus J_{f_\xi}\, d\mu(\xi), \qquad \omega_f \to \int_X^\oplus \omega_{f_\xi}\, d\mu(\xi),$$

$$v_f \to \int_X^\oplus v_{f_\xi}\, d\mu(\xi), \qquad \beta_f \to \int_X^\oplus \beta_{f_\xi}\, d\mu(\xi)$$

(here it is understood that the fields entering the formulas are measurable), and such that the algebra $\mathfrak{Z}_f^{(\beta_f)}$ corresponds to the algebra of diagonal operators in the decomposition $\int_X^\oplus H_{f_\xi}\, d\mu(\xi)$. With this μ we have the formula

$$f(x) = \int_X f_\xi(x)\, d\mu(\xi).$$

Proof. The uniqueness of μ follows at once (bearing in mind that X is standard) from [1, 8.2.4., p. 146] (which shows that another measure μ_1 on X with the same properties must be equivalent to μ) and [2, Theorem 3, p. 203] (which shows that μ and μ_1 must be identical).
Let us then show the existence of such a μ. By [2, Théorème 2, p. 210] there exists a standard Borel space Z, a measure μ on Z, a μ-measurable field $\xi \to H_\xi$ of Hilbert spaces on Z and a unitary carrying H_f onto $\int_Z^\oplus H_{f_\xi}\, d\mu(\xi)$ and $\mathfrak{Z}_f^{(\beta_f)}$ onto the algebra of diagonal operators in this decomposition.

Let $\mathscr{A}'$ be the Hilbert algebra $\mathfrak{n}_f/\mathfrak{n}_f^0$ and let $\mathscr{A}$ be the full Hilbert algebra in H_f defined by $\mathscr{A}'$. By [2, Théorème 1, p. 194] there exists a measurable field $\xi \to \mathscr{A}_\xi$ of full Hilbert algebras, such that $\mathscr{A}_\xi$ is dense in H_ξ and such that $\mathscr{A} = \int_Z^\oplus \mathscr{A}_\xi\, d\mu(\xi)$ (identifying H_f with $\int_Z^\oplus H_\xi\, d\mu(\xi)$). By [2, Proposition 3, p. 190 and Théorème 1, p. 199] we have

$$U_f = \int_Z^{\oplus} U_\xi \, d\mu(\xi), \qquad V_f = \int_Z^{\oplus} V_\xi \, d\mu(\xi),$$

$$J_f = \int_Z^{\oplus} J_\xi \, d\mu(\xi), \qquad \omega_f = \int_Z^{\oplus} \omega_\xi \, d\mu(\xi),$$

where U_ξ, V_ξ, J_ξ, ω_ξ are the usual objects associated with the Hilbert algebra $\mathscr{A}_\xi$. From [2, Théorème 4, p. 176] we have that $\mathfrak{Z}_f = \int_Z^{\oplus} \mathfrak{Z}_\xi \, d\mu(\xi)$, where $\mathfrak{Z}_\xi$ is the center of U_ξ. By [1, 8.3.1., p. 147] there exist measurable fields of representations $\xi \to \lambda_\xi$, $\xi \to \rho_\xi$ of A, A^0, respectively, such that

$$\lambda_f = \int_Z^{\oplus} \lambda_\xi \, d\mu(\xi), \qquad \rho_f = \int_Z^{\oplus} \rho_\xi \, d\mu(\xi).$$

Let x_n, $n \in \mathbb{N}$, be a dense sequence in A. Since $\lambda_f(x_n)$ generates U_f, $\lambda_\xi(x_n)$ generates U_ξ for almost all ξ [2, Théorème 1, p. 171] and therefore $\lambda_\xi(A)$ generates U_ξ for almost all $\xi \in Z$. Similarly for ρ_ξ.

The representation v_f commutes with $\mathfrak{Z}_f^{(\beta_f)}$ and therefore there exists a measurable field of representations $\xi \to v_\xi$ of G on H_ξ, such that

$$v_f(s) = \int_Z^{\oplus} v_\xi(s) \, d\mu(\xi)$$

[1, 8.3.1., p. 147]. Now $v_f(s)\,\lambda_f(x)\,v_f(s^{-1}) = \lambda_f(\alpha_s(x))$ for all $s \in G$, $x \in A$. Therefore, $v_\xi(s)\,\lambda_\xi(x)\,v_\xi(s^{-1}) = \lambda_\xi(\alpha_s(x))$ for almost all ξ, when s, x are fixed, and therefore for almost all ξ when x, s runs through countable dense subsets of A, G, respectively. But then $\lambda_\xi(\alpha_s(x)) = v_\xi(s)\,\lambda_\xi(x)\,v_\xi(s^{-1})$ for all $s \in G$, $x \in A$ for almost all ξ. In particular λ_ξ is G-quasi-invariant for almost all $\xi \in Z$. Let β_ξ be the automorphism group of U_ξ implemented by v_ξ. We then have

$$\beta_f(s) = \int_Z^{\oplus} \beta_\xi(s) \, d\mu(\xi).$$

Since $\mathfrak{Z}_f = \int_Z^{\oplus} \mathfrak{Z}_\xi \, d\mu(\xi)$ and since $\mathfrak{Z}_f^{(\beta_f)}$ is the algebra of diagonal operators we have that $\mathfrak{Z}_\xi^{(\beta_f)} = \mathbb{C} I_{H_\xi}$ for almost all $\xi \in Z$, and therefore λ_ξ is G-factorial for almost all $\xi \in Z$. We also have that

$$\chi(s)\ \omega_f = \omega_f \circ \beta_f(s) = \int_Z^{\oplus} \omega_\xi \circ \beta_\xi(s) \, d\mu(\xi),$$

from which $\omega_\xi \circ \beta_\xi(s) = \chi(s)\ \omega_\xi$ for almost all ξ, s fixed. Letting s run through a countable, dense subset in G and using the σ-weak lower semicontinuity of ω_ξ we find that $\omega_\xi \circ \beta_\xi(s) = \chi(s)\,\omega_\xi$ for all $s \in G$ for almost all $\xi \in Z$.

Let y_n, $n \in \mathbb{N}$, be a sequence in $\mathfrak{n}_f$ such that $\dot{y}_n$ is dense in H_f. Since $\lambda_f(y_n)$ generates U_f, $\lambda_\xi(y_n)$ generates U_ξ for almost all $\xi \in Z$. Moreover,

$$\int_Z \omega_\xi(\lambda_\xi(y_n)^*\lambda_\xi(y_n))\,d\mu(\xi) = \omega_f(\lambda_f(y_n)^*\lambda_f(y_n))$$
$$= f(y_n^* y_n) < +\infty,$$

and therefore $\omega_\xi(\lambda_\xi(y_n) * \lambda_\xi(y_n)) < +\infty$ for all $n \in \mathbb{N}$ for almost all $\xi \in Z$. But this shows that λ_ξ is (G,χ)-normal for almost all $\xi \in Z$ (Definition 1.2.2). Set $f'_\xi = \omega_\xi \circ \lambda_\xi | A^+$, so that f'_ξ is a Borel field of (G,χ)-characters. By removing a null set from Z we get an injective Borel map $\xi \to f'_\xi$ from a Borel subset Z' into $(G, A, \alpha)^\wedge_\chi$. Therefore, we can assume that $Z = (A, G, \alpha)^\wedge_\chi$ and by adjusting the measure μ we can assume that $f'_\xi = f_\xi$ for all $\xi \in Z$ (cf. Proposition 1.9.1). The only thing left to show is then that there exists for almost all ξ a unitary carrying H_ξ to H_{f_ξ}, λ_ξ to λ_{f_ξ}, etc. This can be done precisely as in the proof of [1, 8.8.2 Théorème, pp. 162–163], a proof which we have also otherwise benefited from.

3. The χ-Normal Representations

Let G be a separable locally compact group and let $\chi: G \to \mathbb{R}_+^*$ be a continuous homomorphism.

3.1. Recall that if π is a representation of G, then a χ-relatively invariant weight on $\mathfrak{A} = \pi(G)''$ is a faithful, normal, semifinite weight ω, such that $\omega(\pi(s)\,T\pi(s^{-1})) = \chi(s)\,\omega(T)$, $T \in \mathfrak{A}^+$, $s \in G$ [12, Definition 2.1.1].

Definition 3.1.1. A representation π of G is called χ-normal if

(i) π is G-factorial,

(ii) $\mathfrak{A} = \pi(G)''$ admits a χ-relatively invariant weight,

(iii) there exists $x \in C^*(G)^+$, such that

$$0 < \omega(\pi(x)) < +\infty.$$

(There is essentially one such choice of ω, cf. [12, Proposition 2.1.2].)

If π is a χ-normal representation of G and if ω is a weight on $\mathfrak{A} = \pi(G)''$ as in Definition 3.1.1, then (π, ω) is quasi-equivalent to (λ_f, ω_f) [12, Theorem 2.1.14], where f is the χ-semicharacter $\omega \circ \pi \mid C^*(G)^+$ [12, Theorem 2.1.6].

If π is a χ-normal representation of G and if π' is quasi-equivalent to π, then π' is χ-normal.

We define $\hat{G}_\chi$ to be the set of equivalence classes (under "quasi-equivalence") of χ-normal representations of G. If $\chi \equiv 1$, then $\hat{G}_\chi$ is the set of quasi-equivalence classes of normal representations of G, $\hat{G}_{\mathrm{norm}}$, in the usual sense.

3.2. Set $G_0 = \ker \chi$. We consider the C^*-dynamical system $(C^*(G_0), G, \alpha)$. Here α is the natural action of G in $C^*(G_0)$.

For brevity we shall set $\hat{G}_\chi^0 = (C^*(G_0), G, \alpha)\hat{}_\chi$, the set of quasi-equivalence classes of (G, χ)-normal representations of G_0 (cf. 1.3).

LEMMA 3.2.1. *The map $\xi \to \mathrm{ind}_{G_0 \uparrow G}\, \xi$ sets up a bijection between $\hat{G}_\chi^0$ and $\hat{G}_\chi$.*

Proof. Let $\xi \in \hat{G}_\chi^0$ be represented by λ_f, where f is a (G, χ)-character on G_0. Then $\mathrm{ind}_{G_0 \uparrow G}\, \xi$ is represented by $\mathrm{ind}_{G_0 \uparrow G}\, \lambda_f$, which is equivalent to $\lambda_{\tilde{f}}$, where $\tilde{f}$ is the χ-semicharacter $\mathrm{ind}_{G_0 \uparrow G}\, f$ [14, Definition 2.1.2, Theorem 2.3.1, and Theorem 3.3.1]. This shows that $\xi \to \mathrm{ind}_{G_0 \uparrow G}\, \xi$ maps $\hat{G}_\chi^0$ injectively into $\hat{G}_\chi$. The surjectivity follows from the fact that any χ-semicharacter f on G has the form $\mathrm{ind}_{G_0 \uparrow G}\, f$ for a (G, χ)-character on G_0 [14, Theorem 3.3.1]. This proves the lemma.

THEOREM 3.2.2. *The set $\hat{G}_\chi$ is a Borel subset of $\hat{G}$ when the latter carries the Mackey–Borel structure, and $\hat{G}_\chi$ is a standard Borel space in the induced Borel structure. Moreover, the map $\xi \to \mathrm{ind}_{G_0 \uparrow G}\, \xi \colon \hat{G}_\chi^0 \to \hat{G}_\chi$ is a Borel isomorphism.*

Proof. There exists a Borel function $\xi \to L_\xi \colon \hat{G}_\chi^0 \to \mathrm{Rep}(G_0)$, such that $L_\xi \in \xi$ for all $\xi \in \hat{G}_\chi^0$ (Proposition 1.9.2). Moreover, there exists a Borel function $\xi \to \tilde{L}_\xi \colon \hat{G}_\chi^0 \to \mathrm{Rep}(G)$, such that $\tilde{L}_\xi$ is equivalent to $\mathrm{ind}_{G_0 \uparrow G}\, L_\xi$ for all $\xi \in \hat{G}_\chi^0$ (cf. [9]). By Lemma 3.2.1 the map $\xi \to \tilde{L}_\xi$ is injective and since $\mathrm{Rep}(G)$ is a standard Borel space, $\xi \to \tilde{L}_\xi$ is a Borel isomorphism from $\hat{G}_\chi^0$ onto its image C in $\mathrm{Rep}(G)$, and C is a Borel subset. Now C is in fact a Borel subset of $\mathrm{Fac}(G)$ and, by Lemma 3.2.1, C meets each quasi-equivalence class in $\mathrm{Fac}(G)$ in at most one point. It follows from [1, 7.2.3 Proposition] that the restriction to C of the canonical map ψ from $\mathrm{Fac}(G)$ to C is a Borel isomorphism from C onto its image and that this image is a Borel subset of $\hat{G}$. From Lemma 3.2.1 it follows that this image is precisely $\hat{G}_\chi$, and since $\mathrm{ind}_{G_0 \uparrow G}\, \xi = \psi(\tilde{L}_\xi)$, we have proved the theorem.

In the course of the proof of Theorem 3.2.2 we have proved:

PROPOSITION 3.2.3. *There exists a Borel map $\xi \to L_\xi \colon \hat{G}_\chi \to \mathrm{Fac}(G)$, such that L_ξ is equivalent to λ_f, where f is a χ-semicharacter with $\lambda_f \in \xi$.*

3.3. Let N be a closed, normal subgroup of G contained in $G_0 = \ker \chi$. We shall briefly investigate what happens when we induce a Borel field of (G, χ)-traces on N. First, let f be a (G, χ)-trace on N and let $\tilde{f} = \tilde{\omega}_f \circ \tilde{\lambda}_f \mid C^*(G)^+$ be the induced χ-semitrace (cf. [14, 2.1], assuming for simplicity that G/N is unimodular). Let u_n be a sequence in $\mathfrak{n}_f$ with $\|\lambda_f(u_n)\| \leqslant 1$ and $\lambda_f(u_n) \to I$ strongly. Also, let φ_k be a sequence in $\mathscr{K}(G)$ with $\varphi_k \geqslant 0$, $\int_G \varphi_k(s)\, ds = 1$ and $\operatorname{supp} \varphi_k \subset V_k$, where $k \to V_k$ is a decreasing sequence of neighborhoods of the identity in G, such that $\bigcap_k V_k = \{e\}$. Let ψ_k be the element in $\mathscr{B}$ (cf. [14, 1.2]) given by $\psi_k(s)(n) = \varphi_k(sn)$, $s \in G$, $n \in \mathbb{N}$ and define $\overline{\psi_k}(s) = \psi_k(s)\, u_k$. Then $\overline{\psi_k} \in \mathscr{B}$ and $\overline{\psi_k}$ determines an element $x_k \in C^*(G)$ (cf. loc. cit.). It is easily seen that $\tilde{\lambda}_f(x_k) \to I$ strongly. Also, $\tilde{\omega}_f(\tilde{\lambda}_f(x_k)^* \tilde{\lambda}_f(x_k)) = f(u_n^* u_n) \int_{G/N} \|\psi(s)\|^2\, ds < +\infty$, so $\tilde{\lambda}_f(x_k) \in \mathfrak{n}_{\tilde{\omega}_f}$.

LEMMA 3.3.1. *For $T \in \mathfrak{A}^+$ we have*

$$\tilde{\omega}_f(\tilde{\lambda}_f(x_k)^*\, T \tilde{\lambda}_f(x_k)) \to \tilde{\omega}_f(T).$$

Proof. Let $T \in \mathfrak{A}^+$ and assume first that $T \in \mathfrak{m}_{\tilde{\omega}_f}$. Then $\tilde{\omega}_f(\tilde{\lambda}_f(x_k) * T^{1/2} T^{1/2} \tilde{\lambda}_f(x_k)) = \|(T^{1/2} \tilde{\lambda}_f(x_k))^{\cdot}\|^2$ (norm in $\mathfrak{n}_{\tilde{\omega}_f} \subset H_{\tilde{\omega}_f}$) $= \|S_{\tilde{\omega}_f} l(\tilde{\lambda}_f(x_k))\, S_{\tilde{\omega}_f} \dot{T}^{1/2}\|^2$ $(l(\tilde{\lambda}_f(x_k)) =$ multiplication by $\tilde{\lambda}_f(x_k)$ from the left-hand side$)$ $= \|J_{\tilde{\omega}_f} \Delta_{\tilde{\omega}_f}^{1/2} l(\tilde{\lambda}_f(x_k))\, \Delta_{\tilde{\omega}_f}^{-1/2} J_{\tilde{\omega}_f} \dot{T}^{1/2}\|^2 = \|l(\tilde{\lambda}_f(y_k))\, J_{\tilde{\omega}_f} \dot{T}^{1/2}\|^2$.

Here y_k is the element in $C^*(G)$ which corresponds to the element $s \to \chi(s)^{1/2}\, \overline{\psi_k}(s)$ in $\mathscr{B}$. It is easily seen that $\tilde{\lambda}_f(y_k) \to I$, strongly, and therefore the last expression converges to

$$\|J_{\tilde{\omega}_f} \dot{T}^{1/2}\|^2 = \|\dot{T}^{1/2}\|^2 = \tilde{\omega}_f(T).$$

This shows the lemma when $T \in \mathfrak{m}_{\tilde{\omega}_f}^+$. If $T \notin \mathfrak{m}_{\tilde{\omega}_f}$, we have $\liminf \tilde{\omega}_f(\tilde{\lambda}_f(x_k)^*\, T \tilde{\lambda}_f(x_k)) \geqslant \tilde{\omega}_f(T) = +\infty$ and thus the lemma is valid also in this case. This ends the proof of the lemma.

Let X be a Borel space and let $\xi \to f_\xi$ be a Borel field of (G, χ)-traces on N.

PROPOSITION 3.2.2. *The family $\xi \to \tilde{f}_\xi = \operatorname{ind}_{N \uparrow G} f_\xi$ is a Borel field of χ-semitraces on G.*

Proof. Let S be a countable subset of $A^+ = C^*(N)^+$ as in Lemma 1.5.1. For $a \in S$ set $X_a = \{\xi \in X \mid 0 < f_\xi(a) < +\infty\}$. Then X_a is a Borel subset of X and since $X = \bigcup_{a \in S} X_a$ and $\{X_a \mid a \in S\}$ is countable, it suffices to show that $\xi \to f_\xi(x)$ is a Borel function on X_a for each $x \in C^*(G)^+$ and each $a \in S$. Let $a \in S$ be fixed and let $\mathfrak{A}$ be the α-invariant two-sided ideal in A generated by a. Choose an approximate unit u_n for $\bar{\mathfrak{A}}$ contained in $\mathfrak{A}$. Then $u_n \in \mathfrak{n}_{f_\xi}$ for $\xi \in X_a$ and $\lambda_{f_\xi}(u_n) \to I$ strongly. Using the notation from Lemma 3.3.1 we find that $\tilde{f}_\xi(x_k^* y x_k) \to \tilde{f}_\xi(y)$ for all $\xi \in X$, $y \in C^*(G)^+$. It therefore suffices to show that $\xi \to \tilde{f}_\xi(x_k^* y x_k)$ is a Borel function for each k and $y \in C^*(G)^+$, and for this it suffices to consider y's of the form z^*z, where z arises from an element in $\mathscr{B}$ (cf. above). But this is clear, since $\tilde{f}_\xi(x_k^* z^* z x_k) = f_\xi(u_k^* u_k) \int_{G/N} \|\psi(s)\|^2 \, d\dot{s}$ for some $\psi \in \mathscr{B}$ (cf. proof of [14, Proposition 2.1.1]).

LEMMA 3.3.3. *There exists a countable subset S of $C^*(G)^+$ with the following property: For each χ-semitrace f on G there exists $x \in S$, such that $0 < f(x) < +\infty$.*

Proof. This is easily proved using a subset of $A^+ = C^*(G_0)^+$ as described in Lemma 1.5.1 and arguing as in the proof of [14, Proposition 2.1.1].

PROPOSITION 3.3.4. *There exists a Borel field $\xi \to f_\xi$ of χ-semicharacters on G such that $[f_\xi] = \xi$ for all $\xi \in \hat{G}_\chi$. All other such fields $\xi \to f'_\xi$ have the form $f'_\xi = k(\xi) f_\xi$ for a unique Borel function $k\colon \hat{G}_\chi \to]0, +\infty[$.*

Proof. The existence follows immediately from Proposition 1.9.1, Theorem 3.3.2 and Proposition 3.3.2. The essential uniqueness follows at once from [12, Corollary 2.1.15 and Lemma 3.3.3].

4. DECOMPOSITION OF SEMITRACES

Let G be a separable locally compact group and let $\chi\colon G \to \mathbb{R}_+^*$ be a continuous homomorphism. Let $\xi \to f_\xi$, $\xi \in \hat{G}_\chi$, be a Borel field of χ-semicharacters such that $[f_\xi] = \xi$ for all $\xi \in \hat{G}_\chi$ (Proposition 3.3.4).

THEOREM 4.1. *For each χ-semitrace f on G there exists a measure μ on $\hat{G}_\chi$ uniquely determined by the following property: The field $\xi \to H_{f_\xi}$ can be given a μ-measurable structure such that there exists an isomorphism from H_f onto $\int_{\hat{G}_\chi}^{\oplus} H_{f_\xi} \, d\mu(\xi)$ setting up the following correspondence:*

$$\lambda_f \to \int_{\hat{G}_\chi}^{\oplus} \lambda_{f_\xi} \, d\mu(\xi), \qquad \rho_f \to \int_{\hat{G}_\chi}^{\oplus} \rho_{f_\xi} \, d\mu(\xi),$$

$$U_f \to \int_{\hat{G}_\chi}^{\oplus} U_{f_\xi} \, d\mu(\xi), \qquad V_f \to \int_{\hat{G}_\chi}^{\oplus} V_{f_\xi} \, d\mu(\xi),$$

$$J_f \to \int_{\hat{G}_\chi}^{\oplus} J_{f_\xi} \, d\mu(\xi), \qquad \Delta_f^{it} \to \int_{\hat{G}_\chi}^{\oplus} \Delta_{f_\xi}^{it} \, d\mu(\xi),$$

$$\omega_f \to \int_{\hat{G}_\chi}^{\oplus} \omega_{f_\xi} \, d\mu(\xi)$$

(*here it is understood that the fields entering the formulas are measurable*), *and such that the center* $\mathfrak{Z}_f = U_f \cap V_f$ *of* U_f *corresponds to the algebra of diagonal operators in the decomposition* $\int_{\hat{G}_\chi}^{\oplus} H_{f_\xi}\, d\mu(\xi)$. *With this* μ *we have the formula*

$$f(x) = \int_{\hat{G}_\chi}^{\oplus} f_\xi(x)\, d\mu(\xi)$$

for all $x \in C(G)^+$.

Proof. Let f^0 be the (G, χ)-trace on $G_0 = \ker \chi$ with $\operatorname{ind}_{G_0 \uparrow G} f^0 = f$ and let f_ξ^0 be the (G, χ)-character on G_0 with $\operatorname{ind}_{G_0 \uparrow G} f_\xi^0 = f_\xi$. Then $\xi \to f_\xi^0$ is a Borel field of (G, χ)-characters. We identify $X = (C^*(G_0), G, \alpha)\hat{}_\chi$ and $\hat{G}_\chi$ via the Borel isomorphism described in Theorem 3.2.2. In this fashion $\xi \to f_\xi^0$ is a Borel field of (G, χ)-characters with $[f_\xi^0] = \xi$. Let then μ be the measure on X corresponding to the (G, χ)-trace f^0 and the Borel field $\xi \to f_\xi^0$ as described in Theorem 2.1. We write $H_{f^0} = \int_X^{\oplus} H_{f_\xi^0}\, d\mu(\xi)$, etc.

Set $\tilde{H}_{f^0} = L^2(G, \lambda_{f^0})$ and $\tilde{H}_{f_\xi^0} = L^2(G, \lambda_{f_\xi^0})$ (the spaces associated with the induced representation of λ_{f^0} and $\lambda_{f_\xi^0}$) and let $\sigma: G/G_0 \to G$ be a Borel section with $\sigma(G_0) = \{e\}$. We then have isomorphisms $V_{f^0}^\sigma: L^2(G, \lambda_{f^0}) \to L^2(G/G_0, H_{f^0})$, $V_{f_\xi^0}^\sigma: L^2(G, \lambda_{f_\xi^0}) \to L^2(G/G_0, H_{f_\xi^0})$ given by

$$V_{f^0}^\sigma \varphi(\dot{s}) = \lambda_{f^0}(a(s))\, \varphi(s),$$

where $a(s) = \sigma(\dot{s})^{-1} s$, and similarly for $V_{f_\xi^0}^\sigma$.

Now the field $\xi \to L^2(G/G_0, H_{f_\xi^0}) \simeq L^2(G/G_0) \otimes H_{f_\xi^0}$ can be given a natural μ-measurable structure [2, Proposition 10, p. 151] and in this way $L^2(G/G_0, H_{f^0})$ can be identified with $\int_X^{\oplus} L^2(G/G_0, H_{f_\xi^0})\, d\mu(\xi)$ in a natural way [2, Proposition 11, p. 152]. We then transport the measurable structure on the field $\xi \to L^2(G/G_0, H_{f_\xi^0})$ to the field $\xi \to \tilde{H}_{f_\xi^0}$ via the field $\xi \to V_{f_\xi^0}^\sigma$ of isomorphisms. In this way we have an isomorphism from $\tilde{H}_{f^0}$ to $\int_X^{\oplus} \tilde{H}_{f_\xi^0}\, d\mu(\xi)$. It is easily seen that in this way $\xi \to \tilde{\lambda}_{f_\xi^0} = \operatorname{ind}_{G_0 \uparrow G} \lambda_{f_\xi^0}$ and $\xi \to \tilde{\rho}_{f_\xi^0}$ (cf. [14, 2.3]) are measurable fields of representations and that $\tilde{\lambda}_{f^0}$ corresponds to $\int_X^{\oplus} \tilde{\lambda}_{f_\xi^0}\, d\mu(\xi)$ and $\tilde{\rho}_{f^0}$ corresponds to $\int_X^{\oplus} \tilde{\rho}_{f_\xi^0}\, d\mu(\xi)$. It is also easily seen that $\xi \to \tilde{J}_{f_\xi^0}$ and $\xi \to \Delta_{f_\xi^0}^{it}$ corresponds to $\int_X^{\oplus} \tilde{J}_{f_\xi^0}\, d\mu(\xi)$ and $\int_X^{\oplus} \Delta_{f_\xi^0}^{it}\, d\mu(\xi)$.

For each $\xi \in X$ we have a normal, injective homomorphism $i_{f_\xi^0}: U_{f_\xi^0} \to \tilde{U}_{f_\xi^0}$ characterized by $i_{f_\xi^0}(\lambda_{f_\xi^0}(s)) = \tilde{\lambda}_{f_\xi^0}(s)$, $s \in G_0$. Clearly the field $\xi \to i_{f_\xi^0}$ is measurable and therefore we have an injective, normal homomorphism $\int_X^{\oplus} i_{f_\xi^0}\, d\mu(\xi)$ from $\int_X^{\oplus} U_{f_\xi^0}\, d\mu(\xi)$ into $\int_X^{\oplus} \tilde{U}_{f_\xi^0}\, d\mu(\xi)$. Now clearly $\int_X^{\oplus} i_{f_\xi^0}\, d\mu(\xi)$ maps the diagonal operators in the decomposition $\int_X^{\oplus} U_{f_\xi^0}\, d\mu(\xi)$ into the diagonal operators in the decomposition $\int_X^{\oplus} \tilde{U}_{f_\xi^0}\, d\mu(\xi)$. Moreover, $\int_X^{\oplus} i_{f_\xi}\, d\mu(\xi)(\lambda_{f^0}(s)) = \tilde{\lambda}_{f^0}(s)$, $s \in G_0$, and since $U_{f^0} = \int_X^{\oplus} U_{f_\xi^0}\, d\mu(\xi)$ we therefore have $\tilde{U}_{f^0} = \int_X^{\oplus} \tilde{U}_{f_\xi^0}\, d\mu(\xi)$ and in particular $i_{f^0} = \int_X^{\oplus} i_{f_\xi^0}\, d\mu(\xi)$. Now i_{f^0} maps $\mathfrak{Z}_{f^0}^{(\beta, \rho)}$ into $\mathfrak{Z}_{f^0}$ [14, proof of Theorem 3.3.1] and therefore $\mathfrak{Z}_{f^0}$ corresponds precisely to the diagonal operators in $\int_X^{\oplus} \tilde{U}_{f_\xi^0}\, d\mu(\xi)$. That $\tilde{V}_{f^0} = \int_X^{\oplus} \tilde{V}_{f_\xi^0}\, d\mu(\xi)$ is now a consequence of the results we have already obtained. Also, $\xi \to \tilde{\omega}_{f_\xi^0}$ is a measurable field and $\tilde{\omega}_{f^0} = \int_X^{\oplus} \tilde{\omega}_{f_\xi^0} d\mu(\xi)$. This is easily seen using Lemma 3.3.1 and reasonings from the proof of Proposition 3.3.2.

The theorem now follows from what we have already shown by applying [14, Theorem 2.3.1].

Remark 4.2. Theorem 4.1 applies in particular to the canonical Δ-semitrace f_G on G, where Δ is the modular function on G (that is, f_G is the semitrace associated with the Δ-semitrace class representation (λ_G, ω_G), where λ_G is the left regular representation of G and ω_G is the canonical Δ-relatively invariant weight on $U(G) = \lambda_G(G)''$ derived from the left Hilbert algebra structure on $\mathscr{K}(G)$). In this way we obtain a Plancherel formula for G. Summing up, establishing such a formula for G involves three steps:

(1) Determine the space $\hat{G}_\Delta$ of Δ-normal representations of G.

(2) Determine a Borel field $\xi \to f_\xi$, $\xi \in \hat{G}_\Delta$, of Δ-semicharacters with $|f_\xi| = \xi$ for all $\xi \in \hat{G}_\Delta$.

(3) Find the measure μ determined by f_G as described in Theorem 4.1 [the "Plancherel measure" (associated with $\xi \to f_\xi$)].

We then have, in particular, the formula

$$\|\varphi\|^2 = \int_{\hat{G}_\Delta} f_\xi(\varphi^* * \varphi)\, d\mu(\xi)$$

for all $\varphi \in L^1(G) \cap L^2(G)$. In general this formula does not determine μ. However, there are two important instances where this is the case; namely, let $\xi \to f_\xi$ be a Borel field as in Section 2 and let μ be a measure on $\hat{G}_\Delta$. Then we have:

THEOREM 4.3. *If G is type I or a connected Lie group and if*

$$\|\varphi\|^2 = \int_{\hat{G}_\Delta} f_\xi(\varphi^* * \varphi)\, d\mu(\xi)$$

for all $\varphi \in \mathscr{K}(G)$ (or just $C_c^\infty(G)$ if G is Lie), then μ is the Plancherel measure.

Proof. We only sketch the proof: All the representations λ_{f_ξ} are mutually non-quasi-equivalent and therefore they have distinct kernels in $C^*(G)$ (for type I groups this follows from Glimm's theorem [1, § 9] and for connected Lie groups this follows from the fact that the λ_{f_ξ}'s are actually normal [14] and then by applying Pukanszky's theorem [17, Theorem 1]. But then the decomposition $\int_{\hat{G}_\Delta}^{\oplus} \lambda_{f_\xi} d\mu(\xi)$ is central by a result of Effros [4]. This proves the theorem.

Remark 4.4. The "Plancherel theorem" for non-unimodular groups has been considered by numerous authors [3, 8, 10, 16, 18–20 and others]. The relation between the Plancherel formula for G and for the subgroup $G_0 = \ker \Delta$ has also been investigated in several papers [3, 19, 20 and others]. In our theory we have replaced the space of quasi-orbits of G in $\hat{G}_0$ by the space of (G, χ)-normal representations of G_0. This gives a more natural and manageable theory, in particular in the case where G_0 is not type I. The advantage of this space is also illustrated by Theorem 3.2.2.

EXAMPLE 4.5. We shall demonstrate our program, sketched in Remark 4.2, for setting up a Plancherel formula for a separable locally compact group, with a non-trivial example: Let N be the group $\mathbb{Q} \times \mathbb{R} \times \mathbb{R}$ with the product $(x, y, z)(x', y', z') = (x + x', y + y', z + z' + \frac{1}{2}(xy' - yx'))$ and let $H = \mathbb{Q}_+^*$ act in N by $w \cdot (x, y, z) = (wx, w^{-1}y, z)$. We form the semidirect product $G = N \times_s H$, which is a solvable Lie group. The modular function Δ is given by $\Delta\colon (x, y, z, w) \to w$ so that $\ker \Delta = N$. Let Z be the subgroup $\{(0, 0, z)\}$ of N and let χ_ξ, $\xi \in \mathbb{R}$, be the character on Z given by $\chi_\xi((0, 0, z)) = \exp i\xi z$. Z is the center in G and if π is a factor representation of G, there exists a unique $\xi \in \mathbb{R}$ with $\pi(s) = \chi_\xi(s)I$ for $s \in Z$. χ_ξ is a $(G, 1)$-trace on Z and $\mathrm{ind}_{Z\uparrow N}\, \chi_\xi = f_\xi^0$ is a (G, Δ)-trace on N (cf. [14, 2.1]) and in fact it is easily seen that f_ξ^0 is a (G, Δ)-character on N if $\xi \neq 0$ and that all (G, Δ)-characters on N, which are non-trivial on the center, arise in this fashion (using, e.g., [14]). But then $f_\xi = \mathrm{ind}_{N\uparrow G}\, f_\xi^0 = \mathrm{ind}_{Z\uparrow G}\, \chi_\xi$ (cf. [14, Proposition 2.5.1]) is a Δ-semicharacter and all Δ-semicharacters on G, which are non-trivial on Z, has this form. We shall not go into the determination of the set of Δ-normal representations which are trivial on Z (in fact, this set has Plancherel measure 0, cf. below).

Set $\hat{G}'_\Delta$ to be the subset of $\hat{G}_\Delta$ consisting of representations that are non-trivial on Z. Then $\hat{G}'_\Delta$ is a Borel subset of $\hat{G}_\Delta$. Now $\xi \to \chi_\xi$, $\xi \in \mathbb{R} \setminus \{0\}$, is clearly a Borel field of $(G, 1)$-characters on Z and therefore $\xi \to f_\xi = \mathrm{ind}_{Z\uparrow G}\, \chi_\xi$ is a Borel field of Δ-semitraces on G (Proposition 3.2.2). But then $\xi \to [f_\xi]\colon \mathbb{R} \setminus \{0\} \to \hat{G}'_\Delta$ is a Borel isomorphism.

Let us observe that all the elements in $\hat{G}'_\Delta$ are type III_1. This follows at once from [13, Lemma 6.6] and the fact that the (G, Δ)-characters f_ξ^0 are in fact characters. Moreover, all the Δ-semicharacters f_ξ, $\xi \neq 0$, are smooth (since $f_\xi = \mathrm{ind}_{Z\uparrow G}\, \chi_\xi$ and χ_ξ is smooth, cf. [15, Proposition 1.1.3]) and

$$f_\xi(\varphi) = \int_{\mathbb{R}} \varphi(1, 0, 0, z) \exp(i\xi z)\, dz \qquad (*)$$

for all $\varphi \in C_c^\infty(G)$ [15, Proposition 1.1.3].

To determine the Plancherel measure, let us observe that all λ_{f_ξ}, $\xi \neq 0$, are mutually strongly disjoint; in fact, their restrictions to Z have distinct kernels. Therefore the Plancherel measure is determined by the formula

$$\varphi(e) = \int_{\hat{G}_\Delta} f_\xi(\varphi)\, d\mu(\xi),$$

$\varphi \in C_c^\infty(G)$, cf. the proof of Theorem 4.3. From formula (*) we then see that μ is concentrated on $\hat{G}'_\Delta$ and that $d\mu(\xi) = (2\pi)^{-1}\, d\xi$ (identifying $\mathbb{R}\setminus\{0\}$ with $\hat{G}'_\Delta$).

5. The Global Type of a Semitrace

Let G be a separable locally compact group and let $\chi: G \to \mathbb{R}_+^*$ be a continuous homomorphism. Set $G_0 = \ker \chi$. In this paragraph we shall investigate how the global type of a χ-semitrace f can be described in terms of the (G, χ)-trace f_0 on G_0 with $f = \mathrm{ind}_{G_0 \uparrow G} f_0$ (cf. [14, Theorem 3.3.1]) and the action of G in $(G_0)^\wedge_{\mathrm{norm}}$. The action of G in $C^*(G_0)$ is denoted α.

5.1. First we consider the case where $\chi(G)$ is a *closed* subgroup of $\mathbb{R}_+^*$. For $\xi \in (G_0)^\wedge_{\mathrm{norm}}$ we set

$$S_\xi = \{s \in G \mid s\xi = \xi\}$$

and

$$T_\xi = \{t \in \mathbb{R} \mid \chi(s)^{it} = 1 \text{ for all } s \in S_\xi\}.$$

Definition 5.1.1 ($\chi(G)$ closed in $\mathbb{R}_+^*$). Let f_ξ be a character (unique up to a scalar) for an element $\xi \in (G_0)^\wedge_{\mathrm{norm}}$. We say that ξ is a *regular* point if

(1) $f_\xi \circ \alpha_s = \chi(s) f_\xi$ for all $s \in S_\xi$,

(2) the operator H_χ^{it} defined on $L^2(G, \xi)$ and given by $H_\chi^{it}\varphi(s) = \chi(s)^{it}\, \varphi(s)$, $s \in G$, is contained in $\mathrm{ind}_{G_0 \uparrow G}\, \xi(G)''$ *for all* $t \in T_\xi$.

(3) There exists $x \in C^*(G_0)^+$ such that

$$0 < \int_{G/S_\xi} f_\xi(\alpha_s(x))\, \chi(s)^{-1}\, d\dot{s} < +\infty.$$

More precisely we say that a regular point ξ is 0-regular if $\chi(S_\xi) = \{1\}$, 1-regular if $\chi(S_\xi) = \mathbb{R}_+^*$ and λ-regular, $0 < \lambda < 1$, if $\chi(S_\xi) = \{\lambda^n\}_{n \in \mathbb{Z}}$. Set S_{reg} to be the set of regular points and S_{reg}^λ to be the set of λ-regular points $(0 \leqslant \lambda \leqslant 1)$.

PROPOSITION 5.1.2 ($\chi(G)$ closed in $\mathbb{R}_+^*$). *For a given* $0 \leqslant \lambda \leqslant 1$ *the set* S_{reg}^{λ} *of* λ*-regular points is a Borel subset of* $(G_0)^{\hat{}}_{\text{norm}}$.

Proof. Let G_1 be the common stabilizer S_ξ for all λ-regular points ξ and $T = \{t \in \mathbb{R} \mid \chi(s)^{it} = 1 \text{ for all } s \in G_1\}$. First we prove:

LEMMA 5.1.3. *If* $\xi \in S_{\text{reg}}^{\lambda}$, *then* $\operatorname{ind}_{G_0 \uparrow G} \xi(G)'' = \mathfrak{A}$ *admits a* χ*-relatively invariant weight.*

Proof. Set $\chi_1 = \chi \mid G_1$. Then f_ξ is a (G_1, χ_1)-character and therefore $\operatorname{ind}_{G_0 \uparrow G_1} \xi = \xi_1$ is (G_1, χ_1)-normal. In particular $\xi_1(G_1)'' = \mathfrak{A}_1$ admits a χ_1-relatively invariant weight. Now $\tilde{\xi} = \operatorname{ind}_{G_0 \uparrow G} \xi$ is equivalent to $\operatorname{ind}_{G_1 \uparrow G} \xi_1$ and in this picture the operators H_χ^{it}, $t \in T$, are expressed by $H_\chi^{it} \varphi(s) = \chi(s)^{it} \varphi(s)$ ($\varphi \in L^2(G, \xi_1)$). Since $\dot{s} \to \chi(s)^{it}$ defines a character on G/G_1 for each $t \in T$ and since these characters separate points in G/G_1, it follows that the von Neumann algebra $\mathfrak{A} = \tilde{\xi}(G)''$ is identical to the von Neumann algebra generated by the canonical imprimitivity system associated with $\operatorname{ind}_{G_1 \uparrow G} \xi_1$ (because H_χ^{it}, $t \in T$, is in $\operatorname{ind}_{G_1 \uparrow G} \xi_1(G)''$ by condition 2 in Definition 5.1.1). But this latter von Neumann algebra can be identified with $\mathfrak{A}_1 \otimes B(L^2(G/G_1, d\dot{s}))$ (after a Borel section $\sigma: G/G_1 \to G_+^*$ has been selected). The weight $\omega = \omega_1 \otimes \operatorname{Tr}(A \,\cdot\,)$, where A is the operator on $L^2(G/G_1, d\dot{s})$ given by $A\psi(\dot{s}) = \chi(\sigma(\dot{s}))\, \psi(\dot{s})$, on $\mathfrak{A}$ is easily seen to be χ-relatively invariant. This proves the lemma.

LEMMA 5.1.4. *For* $\xi \in S_{\text{reg}}^{\lambda}$ *the decomposition*

$$\int_{G/G_1}^{\oplus} \lambda_{f_\xi} \circ \alpha_{\sigma(\dot{s})}^{-1} \, d\dot{s}$$

is central ($\sigma: G/G_1 \to G$ *a Borel section*).

Proof. In fact, $\mathfrak{A}$ admits a χ-relatively invariant weight ω (Lemma 5.1.3) from which H_χ^{it}, $t \in T$, implements the modular group σ^ω at the point t. But this implies that H_χ^{it} is fixed by σ_t^ω and therefore H_χ^{it} is actually in the von Neumann algebra generated by $\tilde{\xi}(s)$, $s \in G_1$. Now $\tilde{\xi} \mid G_1$ is equivalent to

$$\int_{G/G_1}^{\oplus} \lambda_{f_1} \circ \alpha_{\sigma(\dot{s})}^{-1} \, d\dot{s}, \tag{*}$$

where $f_1 = \operatorname{ind}_{G_0 \uparrow G_1} f_\xi$ and α also denotes the action of G in $C^*(G_1)$, and in this decomposition H_χ^{it} is expressed as

$$\int_{G/G_1}^{\oplus} \chi(\dot{s})^{it}\, I \, ds,$$

and therefore the decomposition (*) is central. But then the center in (*) is actually generated by $\tilde{\xi}$'s restriction to G_0, which is equivalent to

$$\int_{G/G_1}^{\oplus} \lambda_{f_\xi} \circ \alpha_{\sigma(\dot{s})}^{-1}\, d\dot{s},$$

and it follows that this decomposition is central.

LEMMA 5.1.5. *The formula*

$$\tilde{f}_\xi(x) = \int_{G/G_1} f_\xi \circ \alpha_s(x)\, \chi(s)^{-1}\, d\dot{s}$$

defines a (G, χ)-character on G_0.

Proof. In fact, this follows from condition 3 in Definition 5.1.1 and Lemma 5.1.4.

Proof of Proposition 5.1.2. Let S be a countable subset of $C^*(G_0)^+$ as described in Lemma 1.5.1. We can assume that S has the property that if two lower semicontinuous, semifinite traces on $C^*(G_0)$ agree on S, then they are equal; in fact, the subset S constructed in the proof of Lemma 1.5.1 is easily seen to have this property. Also, let G_0' be a countable, dense subset in G_0 and let T' be a countable, dense subset in T. Finally, let $\xi \to f_\xi$, $\xi \in (G_0)\hat{}_{\text{norm}}$, be a Borel field of characters on G with $[f_\xi] = \xi$. Then $E_1 = \{\xi \mid f_\xi \circ \alpha_s = \chi(s) f_\xi \quad \text{for all} \quad s \in G_1\} = \bigcap_{x \in S} \bigcap_{s \in G_0'} \{\xi \mid f_\xi \circ \alpha_s(x) = \chi(s) f_\xi(x)\}$ is a Borel subset of $(G_0)\hat{}_{\text{norm}}$.

Let then $\xi \to L_\xi \colon (G_0)\hat{}_{\text{norm}} \to \text{Rep}(G_0)$ be a Borel map such that $L_\xi \simeq \lambda_{f_\xi}$ (Proposition 1.9.2) and set $B_n = \{\xi \mid \dim L_\xi = n\}$. Let H_n be a Hilbert space of dimension n. Proceeding as in the proof of Theorem 4.1 we can realize $\text{ind}_{G_0 \uparrow G} L_\xi = \tilde{L}_\xi$ on $L_2(G/G_0, H_n)$ and in this manner H_χ^{it} is realized as $H_\chi^{it}\varphi(\dot{s}) = \chi(s)^{it}\, \varphi(\dot{s})$ for $\xi \in B_n$, and $\xi \to \tilde{L}_\xi$ is a Borel map. Since $H^{it_n} \to^s H^{it}$ for $t_n \to t$, it follows that

$$\{\xi \in B_n \mid H_\chi^{it} \in \text{ind}_{G_0 \uparrow G} L_\xi(G)''\}$$

$$= \bigcap_{t \in T'} \{\xi \in B_n \mid H_\chi^{it} \in \text{ind}_{G_0 \uparrow G} L_\xi(G)''\},$$

which is easily seen to be a Borel set. It follows that the set of $\xi \in (G_0)_{\text{norm}}$ satisfying condition 2 in Definition 5.1.1 is a Borel subset. Set E_2 to be the set of $\xi \in (G_0)\hat{}_{\text{norm}}$ satisfying conditions 1 and 2. We have seen that E_2 is a Borel subset. Now using Lemma 5.1.5 and the properties of the set $S \subset C^*(G_0)^+$ we see that S_{reg}^λ is equal to

$$\bigcap_{x \in S} \left\{ \xi \in E_2 \mid 0 < \int_{G/G_1} f_\xi \circ \alpha_s(x) \chi(s)^{-1} \, d\dot{s} < +\infty \right\},$$

which is a Borel subset. This shows the proposition.

We saw in Lemma 5.1.5 that for $\xi \in S^\lambda_{\text{reg}}$ the formula $\tilde{f}_\xi(x) = \int_{G/G_1} f_\xi(\alpha_s(x)) \chi(s)^{-1} \, d\dot{s}$ defines a (G, χ)-character on G_0, and clearly $\xi \to \tilde{f}_\xi$ is a Borel field of (G, χ)-characters. Therefore, we have a Borel map $p: \xi \to [f_\xi]: S^\lambda_{\text{reg}} \to (C^*(G_0), G, \alpha)^\wedge_\chi$ and $p^{-1}(p(\xi)) = G\xi$. Using this map we prove:

PROPOSITION 5.1.6 ($\chi(G)$ closed in $\mathbb{R}^*_+$). *The orbit space S^λ_{reg}/G admits a Borel transversal (and in particular it is standard).*

Proof. Let $\eta \to F_\eta$, $\eta \in (C^*(G_0), G, \alpha)^\wedge_\chi$ be a Borel field of (G, χ)-characters with $\eta = [F_\eta]$ (Proposition 1.9.1). If $\chi(G) = \mathbb{R}^*_+$, the set $\{\xi \in S^\lambda_{\text{reg}} \mid \tilde{f}_\xi = F_{p(\xi)}\}$ is a Borel transversal. If $\chi(G)$ is isomorphic to $\mathbb{Z}$, the same proof works with trivial modifications.

THEOREM 5.1.7 ($\chi(G)$ closed in $\mathbb{R}^*_+$). *Let f be a χ-semitrace and let f_0 be the (G, χ)-trace with $\operatorname{ind}_{G_0 \uparrow G} f_0 = f$. If μ_0 is the canonical measure for f_0, then*

(i) *f is type I if and only if μ_0 is concentrated on $(G_0)^\wedge_{\text{norm}}$ (i.e., if and only if f_0 is type I).*

(ii) *f is semifinite if and only if μ_0 is concentrated on S^0_{reg}.*

(iii) *f is type III_λ, $0 < \lambda \leqslant 1$, if and only if μ_0 is concentrated on S^λ_{reg}.*

Proof. This is easily seen using the two decompositions of f_0 into (G, χ)-characters and characters, respectively, exploiting the existence of a Borel transversal in S^λ_{reg}/G and then finally invoking [13, Theorems 6.2 and 6.11 and Proposition 6.5].

5.2. Now we put no restrictions on $\chi(G)$. On the other hand we consider only the type I part of $(G_0)^\wedge_{\text{norm}}$:

DEFINITION 5.2.1. An element $\xi \in (G_0)^\wedge_{\text{norm}}$ is said to be *regular* if

(1) The stabilizer of ξ is G_0,

(2) there exists $x \in C^*(G_0)^+$ such that

$$0 < \int_{G/G_0} f_\xi \circ \alpha_s(x) \chi(s)^{-1} \, d\dot{s} < +\infty.$$

This definition is easily seen to coincide with Definition 5.1.1 when $\chi(G)$ is closed (which was the only case considered in 5.1). We set S^1_{reg} to be set of type I regular points. We have the following analog of Proposition 5.1.2:

PROPOSITION 5.2.2. *The set* S^1_{reg} *is a Borel subset of* $(G_0)^{\wedge}_{\text{norm}}$.

Proof. This is proved along the same lines of reasoning as Proposition 5.1.1. Note that here the decomposition

$$\int_{G/G_0}^{\oplus} \lambda_{f_\xi} \circ \alpha_s^{-1} \, d\dot{s}$$

is automatically central, since ξ is GCR [4].

We also have:

PROPOSITION 5.2.3. *The orbit space* S^1_{reg}/G *admits a Borel transversal (and in particular it is standard).*

Proof. We can assume that G/G_0 is discrete (since the case $\chi(G) = \mathbb{R}^*_+$ is covered by Proposition 5.1.6). Denoting by X the subset of elements in $\text{Irr}(G_0)$ satisfying conditions 1 and 2 in Definition 5.2.1, X is a Borel subset, and $G \times U_n$, U_n the unitary group in H_n, acts as a Polish transformation group in $X_n = X \cap \text{Irr}_n(G_0)$. This transformation group satisfies condition D in [5, p. 47] and the resulting orbit space is countably separated. But then it admits a Borel transversal [5, Theorem 2.9, p. 47], and then also S^1_{reg}/G admits a Borel transversal [1, 7.2.3 Proposition, p. 136].

THEOREM 5.2.4. *Let f be a χ-semitrace and let f_0 be the (G, χ)-trace with* $\text{ind}_{G_0 \uparrow G} f_0 = f$. *If μ_0 is the canonical measure for f_0, then f is type* I *if and only if μ_0 is concentrated on* S^1_{reg}.

Proof. Use Proposition 5.2.3 and [13, Theorem 6.4] (cf. proof of Theorem 5.1.7).

Remark 5.2.5. Applying Theorem 5.1.3 to the canonical Δ-semitrace f_G on G we find that $\lambda_G(G)''$ is type I if and only if the Plancherel measure for G_0 is concentrated on S^1_{reg}. This result (in connection with Proposition 5.1.2) was obtained by Duflo and Moore [3] in a somewhat weaker form ([3, Corollary 1, p. 240]. The main difference between their result and ours is that we exhibit a canonical Borel subset of $(G_0)^{\wedge}_{\text{norm}}$, namely, S^1_{reg}, such that the action of G in it is smooth and such that the type I part of λ_G can be singled out by means of this subset. Similar remarks apply to Theorem 5.1.7, which (together with Proposition 5.1.6) contain results of Sutherland [19, Theorem 3.4, p. 236 and Theorem 4.4, p. 243].

Remarks 5.2.6. If the set of regular points S_{reg} is a Borel set and $\chi(G)$ is a closed subgroup in $\mathbb{R}_+^*$ (which is the case, e.g., if $\chi(G)$ is isomorphic to $\mathbb{Z}$), then a χ-semitrace f is type III_0 if and only if the canonical measure μ_0 for the (G,χ)-trace f_0 with $\mathrm{ind}_{G_0\uparrow G} f_0 = f$ is concentrated on the (Borel) set of irregular points. I was not able to prove in full generality that the set of all regular points is a Borel set.

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